Homological algebra solutions Week 4

1. Consider the commutative diagram of R-modules

$$\begin{tikzcd} & A' \arrow[r, "i'"] \arrow[d, "f"] & B' \arrow[r, "p'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & C & \end{tikzcd}$$

We want to show that the sequence

$$\ker(f) \xrightarrow{\alpha} \ker(g) \xrightarrow{\beta} \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \xrightarrow{\varphi} \operatorname{coker}(g) \xrightarrow{\psi} \operatorname{coker}(h)$$

is exact, where α, β, φ, and ψ are respectively induced by i', p', i and p. These morphisms are well defined by commutativity of the diagram. Let $c \in \ker(h)$, by surjectivity of p', there is $b \in B'$ with $p'(b) = c$. By commutativity of the diagram, $pg(b) = hp'(b) = 0$, so $g(b) \in \ker(p) = \operatorname{im}(i)$, thus there exists some $a \in A$ such that $i(a) = g(b)$. In order to define $\partial(c) = a$, we have to verify that this is well defined, the only non-canonical choice we made is the choice of the preimage $b \in B'$. Let $b, b' \in B'$ such that $p'(b) = c = p'(b')$, and a, a' such that $i(a) = g(b), i(a') = g(b')$. Then $b - b' \in \ker(p') = \operatorname{im}(i')$, so there exists $\tilde{a} \in A'$ such that $i'(\tilde{a}) = b - b'$. We have that $g(b - b')) = if(\tilde{a})$, so by injectivity of i, $a - a' = f(\tilde{a}) \in \operatorname{im}(f)$, so $\partial : \ker(h) \to \operatorname{coker}(f)$ is well defined.

We start by showing that $\ker(\beta) = \operatorname{im}(\alpha)$. We know that $p'i' = 0$, thus $\beta\alpha = 0$ so we just have to prove that $\ker(\beta) \subset \operatorname{im}(\alpha)$. Let $x \in \ker(\beta) \subset \ker(g)$, in particular, $x \in \ker(p') = \operatorname{im}(i')$, i.e. there exists some $y \in A'$ such that $i'(y) = x$. We know that $if(y) = gi'(y) = g(x) = 0$, and since i is injective, $y \in \ker(f)$. By definition of α, $\alpha(y) = x$, so the sequence is exact at $\ker(g)$.

We will now show that $\ker(\partial) = \operatorname{im}(\beta)$. Let $x \in \ker(g)$, then

$$\partial\beta(x) = \partial p'(x) = i^{-1}gp'^{-1}p'(x) = i^{-1}g(x) = 0,$$

so $\operatorname{im}(\beta) \subset \ker(\partial)$. Let $x \in \ker(\partial) \subset \ker(h)$, by surjectivity of p', there exists some $y \in B'$ such that $p'(y) = x$. Since $pg(y) = hp'(y) = h(x) = 0$, $y \in \ker(p) = \operatorname{im}(i)$, so there is some $z \in A$ such that $i(z) = g(y)$. By definition of ∂, $z + \operatorname{im}(f) = \partial(x) + \operatorname{im}(f) = 0 + \operatorname{im}(f)$, that means that $z \in \operatorname{im}(f)$, so there exists $a \in A'$ such that $f(a) = z$. Moreover, $g(y - i'(a)) = g(y) - if(a) = g(y) - i(z) = 0$, thus $i'(a) - y \in \ker(g)$ and $\beta(y - i'(a)) = \beta(y) - p'i'(a) = \beta(y) = x$. Therefore, $x \in \operatorname{im}(\beta)$, and we conclude that the sequence is exact at $\ker(h)$.

We prove that $\ker(\varphi) = \operatorname{im}(\partial)$. Let $x \in \ker(h)$, then $\varphi\partial(x) = gp^{-1}(x) = 0 + \operatorname{im}(g)$ i.e. $\operatorname{im}(\partial) \subset \ker(\varphi)$. Now assume $x \in \ker(\varphi)$, and let $\tilde{x} \in A$ be a residue of x, then $i(\tilde{x}) \in \operatorname{im}(g)$, so there is some $y \in B'$ such that $g(y) = i(\tilde{x})$. We see that $hp'(y) = pg(y) = pi(\tilde{x}) = 0$, thus $p'(y) \in \ker(h)$, and by definition $\partial p'(y) = x$, so $x \in \operatorname{im}(\partial)$.

Finally, we prove that the sequence is exact at $\operatorname{coker}(g)$. By exactness of the second row, $\psi\varphi = 0$. Let $x \in \ker\psi$ and $\tilde{x} \in B$ a residue of x, then $p(\tilde{x}) \in \operatorname{im}(h)$, so there is some $y \in C'$ such that $h(y) = p(\tilde{x})$. By surjectivity of p', there also is some $z \in B'$ such that $p'(z) = y$. We can see that $p(\tilde{x}) = h(y) = hp'(z) = pg(z)$, so $\tilde{x} - g(z) \in \ker(p) = \operatorname{im}(i)$. Let $a \in A$ be such that $i(a) = \tilde{x} - g(z)$, then $\varphi(a) = x - g(z) = x + \operatorname{im}(g)$, thus $x \in \operatorname{im}(\varphi)$. We conclude that the sequence is indeed exact.

Moreover, if $i' : A' \to B'$ is injective, its restriction to $\ker(f)$ is also injective. If $p : B \to C$ is surjective, the quotient map $\operatorname{coker}(g) \to \operatorname{coker}(h)$ will also be surjective by definition.

2. (a) By the long exact sequence theorem , there is a long exact sequence

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

Recall that a complex A is exact if and only if $H_n(A) = 0$ for every n. If two of the three complexes are exact, the sequence is of the form

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_n(I) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where $I \in \{A, B, C\}$ depending on our assumption. By exactness of the long sequence, $H_n(I) = 0$ for every n, which implies that the complex I is exact.

(b) The long exact sequence induced by the short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow C \xrightarrow{\alpha} \operatorname{im}(f) \longrightarrow 0$$

shows that α is a quasi-isomorphism, and the long exact sequence induced by the short exact sequence

$$0 \longrightarrow \operatorname{im}(f) \xrightarrow{\beta} D \longrightarrow \operatorname{coker}(f) \longrightarrow 0$$

shows that β is a quasi-isomorphism. Thus $f = \beta \circ \alpha$ is a quasi-isomorphism.

The converse is false. Indeed consider the following morphism of chain maps

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "id"] \arrow[d, "id"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "id"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The rows are exact, so this is a quasi-isomorphism. The complex of the kernel of this morphism is given by

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0\ ,$$

which is not acyclic.

3. We can easily see that the kernel and the image of the morphism $\mathbb{Z}/4\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}/4\mathbb{Z}$ are both $\mathbb{Z}/2\mathbb{Z}$, thus the homology groups are 0 and the complex is acyclic. Let $\varphi : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, the image of $\cdot 2$ is $\{0, 2\}$, $\varphi(2)$ has to be even, and $\cdot 2$ sends even number to 0, so $\cdot 2 \circ \varphi \circ \cdot 2 = 0$, thus this sequence does not split.

4. We show that a chain map $\{s_n : C_{n-1} \to D_n\}$ makes the following diagram commutes i.e. f extends to a map $(-s, f) : \operatorname{cone}(C) \to D$

$$\begin{tikzcd}
\operatorname{cone}(C)_{n+1} \arrow[r, "\delta"] \arrow[d, "{(-s,f)}"] & \operatorname{cone}(C)_n \arrow[d, "{(-s,f)}"] \\
D_{n+1} \arrow[r, "d"] & D_n
\end{tikzcd}$$

if and only if it satisfies $f = ds + sd$ i.e. f is null homotopic. We recall that the differential of $\operatorname{cone}(C)$ is given by $\delta(c_n, c_{n+1}) = (-d(c_n), d(c_{n+1} - c_n))$. Let $c_n \in C_n, c_{n+1} \in C_{n+1}$, then

$$\begin{aligned}
f(c_n) - ds(c_n) - sd(c_n) &= df(c_{n+1}) - fd(c_{n+1}) + f(c_n) - ds(c_n) - sd(c_n) \\
&= d(f(c_{n+1}) - s(c_n)) - (fd(c_{n+1}) - f(c_n) + sd(c_n)) \\
&= (d \circ (-s, f))(c_n, c_{n+1}) - ((-s, f) \circ \delta)(c_n, c_{n+1}).
\end{aligned}$$